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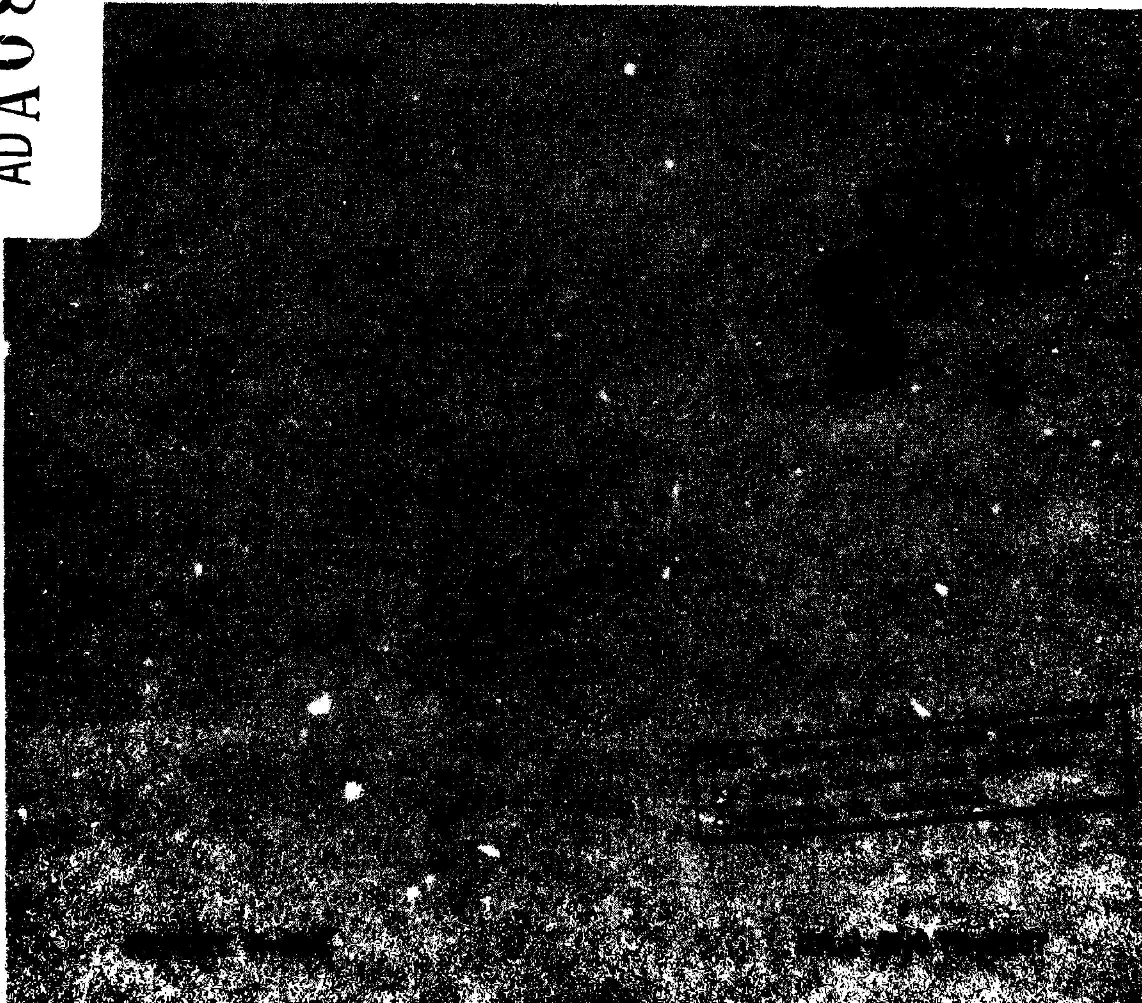
LEVEL II

A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL BIN PACKING ALGORITHMS



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Let $L = (p_1, p_2, \ldots, p_n)$ be a list of real numbers in the interval (0,1]. The one-dimensional bin packing problem is to place the p_i's into a minimum number of unit-capacity bins. For any algorithm A, let A(L) denote the number of bins used by A in packing L and let OPT(L) denote the minimum number of bins needed to pack L. It is shown that, for any on-line algorithm A,

$$\lim_{n\to\infty} \left\{ \max_{OPT(L) = n} \frac{A(L)}{OPT(L)} \right\} > 1.536.$$

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A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL BIN PACKING ALGORITHMS

by

Donna J. Brown

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A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL

BIN PACKING ALGORITHMS†

Donna J. Brown

Coordinated Science Laboratory*
University of Illinois at Urbana

December 1979

Abstract

Let L = (p_1, p_2, ..., p_n) be a list of real numbers in the interval (0, 1]. The one-dimensional bin packing problem is to place the p_i's into a minimum number of unit-capacity bins. For any algorithm A, let A(L) denote the number of bins used by A in packing L and let OPT(L) denote the minimum number of bins needed to pack L. It is shown that, for any on-line algorithm A,

$$\lim_{n\to\infty} \{ \max_{OPT(L)=n} \frac{A(L)}{OPT(L)} \} > 1.536 .$$

†This work was supported by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under contract N00014-79-C-0424.

*Also, Department of Electrical Engineering

I. Introduction

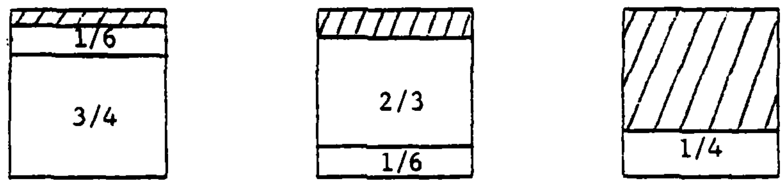
Let $L = (p_1, p_2, \ldots, p_n)$ be a list of real numbers in the interval $(0, 1]$. The <u>one-dimensional bin packing problem</u> is to place the p_i's into a minimum number of unit-capacity bins; i.e., the sum of the numbers in each bin can be at most 1. Because this problem is known to be NP-hard [8], much work has been done in the study of heuristic algorithms with guaranteed performance bounds [12, 13, 14, 16].

~~In~~ this paper is ~~we are~~ concerned with algorithms for which the pieces (numbers) in list L are available one at a time, and each piece must be placed in some bin before the next piece is available; such an algorithm is referred to as <u>on-line</u> [~~12, 13, 16~~]. The performance measure used is the ratio of the number of bins used by an algorithm A in packing list L, A(L), to the optimum (minimum) number of bins required to pack the list, OPT(L).

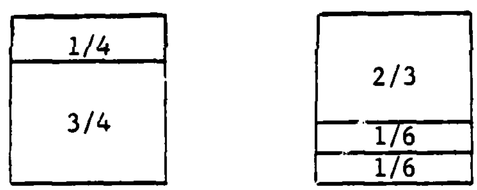
<u>Example 1</u>. Consider the list $L_1 = (3/4, 1/6, 1/6, 2/3, 1/4)$. One possible packing algorithm is the well known First-Fit (FF) Algorithm [12,13,14], which places each piece in the first bin which has enough available space. As shown in Figure 1a, this algorithm leads to a packing which uses three bins. An optimal packing requires only two bins (see Figure 1b). Notice that $FF(L_1) = \frac{3}{2}\, OPT(L_1)$. ∎

We are interested, however, in the ratio $\frac{A(L)}{OPT(L)}$ for lists L with many pieces. In particular, we wish to determine a lower bound on the performance ratio

$$\lim_{n \to \infty} \left\{ \max_{OPT(L) = n} \frac{A(L)}{OPT(L)} \right\}.$$



a) Packing L_1 by the First-Fit Algorithm: $FF(L_1) = 3$.



b) An optimal packing of L_1: $OPT(L_1) = 2$.

Figure 1. Packings of L_1 from Example 1.

Example 2. For n even, let the list L_2 consist of n pieces of size 3/8 and n pieces of size 5/8. The First-Fit Algorithm uses $\frac{3n}{2}$ bins, compared to an optimal packing of n bins (see figures 2a and 2b). Thus, we know that, for the First-Fit Algorithm,

$$FF(L_2) \geq \frac{3}{2}\, OPT(L_2).$$

(In fact, it is known [12,13], that there is a list L for which $FF(L) = \frac{17}{10}\, OPT(L)$.) ∎

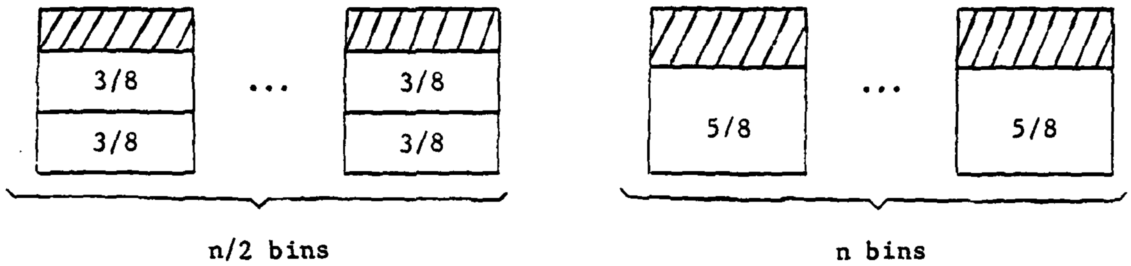
We shall show that there is no algorithm which can always use fewer than 1.536 OPT(L) number of bins. Thus, for any packing algorithm A,

$$\lim_{n \to \infty} \{ \max_{OPT(L) = n} \frac{A(L)}{OPT(L)} \} > 1.536$$

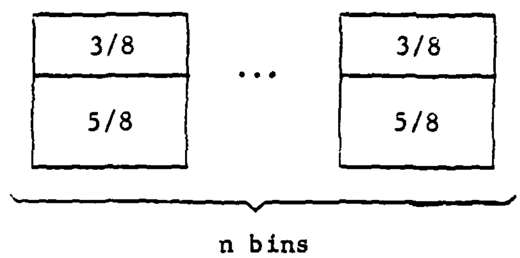
This lower bound is an improvement over the bound of 1.5 proved by Yao [16].

On the upper bound side, Yao in [16] gave an algorithm with a performance ratio of 5/3, an improvement over the 17/10 of the First-Fit Algorithm. Brown [4] has an algorithm with a slightly better performance ratio of about 1.65.

Much work has recently been done with two-dimensional bin packing. Various algorithms [1, 2, 3, 7, 9] have been proposed, many using ideas from one-dimensional packing algorithms [12,13,14]. Some work on two-dimensional lower bounds has also been done [5, 6, 15]. In particular, the 1.536 lower bound presented in this paper extends immediately to two dimensions and gives a 1.536 lower bound for any on-line two-dimensional algorithm which packs pieces in order of decreasing or increasing height or increasing width [6].



a) Packing L_2 by the First-Fit Algorithm: $FF(L_2) = \frac{3n}{2}$.



b) An optimal packing of L_2: $OPT(L_2) = n$.

Figure 2. Packings of L_2 from Example 2.

II. An Example

Yao [16] used a list consisting of pieces of sizes $\frac{1}{6} - 2\varepsilon$, $\frac{1}{3} + \varepsilon$, $\frac{1}{2} + \varepsilon$ in order to obtain his $\frac{3}{2}$ lower bound for any on-line bin packing algorithm. In this section we show that the result can be improved to $\frac{109}{71} > 1.535$ by considering a list with pieces sized $\frac{1}{42} - 3\varepsilon$, $\frac{1}{7} + \varepsilon$, $\frac{1}{3} + \varepsilon$, $\frac{1}{2} + \varepsilon$. In Section III the method is generalized to a list with pieces of t different sizes. The work in this section is therefore only a special case of what will be shown, but it is presented here to illustrate the method and therefore make the proof of the main theorem easier to understand. (Also, $\frac{109}{71}$ is not much smaller than 1.536.)

Let ε be a small positive number, $0 < \varepsilon < \frac{1}{43 \cdot 42 \cdot 3}$. For n a multiple of 42, consider the list $L = L_1 L_2 L_3 L_4$, where

L_1 consists of n pieces of size $\frac{1}{42} - 3\varepsilon$,

L_2 consists of n pieces of size $\frac{1}{7} + \varepsilon$,

L_3 consists of n pieces of size $\frac{1}{3} + \varepsilon$,

L_4 consists of n pieces of size $\frac{1}{2} + \varepsilon$.

Noting that

$$\begin{aligned} OPT(L_1) &= \frac{n}{42}, \\ OPT(L_1 L_2) &= \frac{n}{6}, \\ OPT(L_1 L_2 L_3) &= \frac{n}{2}, \\ OPT(L) &= n, \end{aligned}$$

we can define the ratios

$$\begin{aligned} r_1(n) &= \frac{A(L_1)}{OPT(L_1)} = \frac{42}{n} A(L_1), \\ r_2(n) &= \frac{A(L_1 L_2)}{OPT(L_1 L_2)} = \frac{6}{n} A(L_1 L_2), \end{aligned} \tag{2.1}$$

$$r_3(n) = \frac{A(L_1L_2L_3)}{OPT(L_1L_2L_3)} = \frac{2}{n} A(L_1L_2L_3),$$

$$r_4(n) = \frac{A(L)}{OPT(L)} = \frac{1}{n} A(L) .$$

We shall prove that

$$\max\{r_1(n),\ r_2(n),\ r_3(n),\ r_4(n)\} \geq \frac{109}{71} .$$

Let B denote the set of bins packed by an algorithm A, after the pieces in $L_1\,L_2\,L_3$ have been packed. Each bin $b_w \in B$ $(1 \leq w \leq |B|)$ contains $m_{1,w}$ pieces of size $\frac{1}{42} - 3\epsilon$, $m_{2,w}$ pieces of size $\frac{1}{7} + \epsilon$, and $m_{3,w}$ pieces of size $\frac{1}{3} + \epsilon$. (Note that $m_{1,w}, m_{2,w}$, and $m_{3,w}$ are nonnegative integers, $0 \leq m_{1,w} \leq 42$, $0 \leq m_{2,w} < 7$, $0 \leq m_{3,w} < 3$.) For notational convenience, we shall omit the double subscript and simply write m_j when we mean $m_{j,w}$. We define the set of bins $\alpha_i (1 \leq i \leq 3)$ as follows:

$$\alpha_i = \{b_w \in B \mid b_w \text{ is at least half full, } m_i \neq 0, \text{ and } m_j = 0 \text{ for } 1 \leq j < i\}.$$

In other words, a bin b_w is in

$$\alpha_1 \quad \text{if} \quad \frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 > \frac{1}{2} \quad \text{and} \quad m_1 \neq 0$$
$$\alpha_2 \quad \text{if} \quad \frac{1}{7} m_2 + \frac{1}{3} m_3 > \frac{1}{2} \quad \text{and } m_2 \neq 0,\ m_1 = 0$$
$$\alpha_3 \quad \text{if} \quad \frac{1}{3} m_3 > \frac{1}{2} \quad \text{and} \quad m_3 \neq 0,\ m_1 = m_2 = 0.$$

Similar, we define $\beta_i (1 \leq i \leq 3)$ to be:

$$\beta_i = \{b_w \in B \mid b_w \text{ is less than half full, } m_i \neq 0, \text{ and } m_j = 0 \text{ for } 1 \leq j < i\}.$$

Thus, a bin b_w is in

$$\begin{array}{lll} \beta_1 & \text{if} & \frac{1}{42}m_1 + \frac{1}{7}m_2 + \frac{1}{3}m_3 < \frac{1}{2} \text{ and } m_1 \neq 0 \\ \beta_2 & \text{if} & \frac{1}{7}m_2 + \frac{1}{3}m_3 < \frac{1}{2} \text{ and } m_2 \neq 0,\ m_1 = 0 \\ \beta_3 & \text{if} & \frac{1}{3}m_3 < \frac{1}{2} \text{ and } m_3 \neq 0,\ m_1 = m_2 = 0. \end{array}$$

Letting $|\alpha_i|$ $(|\beta_i|)$ represent the number of bins in α_i (β_i), we have

$$\begin{aligned} A(L_1) &= |\alpha_1| + |\beta_1| \\ A(L_1 L_2) &= |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| \\ A(L_1 L_2 L_3) &= |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\alpha_3| + |\beta_3| \end{aligned} \tag{2.2}$$

Notice that no two pieces of size $\frac{1}{2} + \epsilon$ will fit in the same bin, nor will any of the n pieces of size $\frac{1}{2} + \epsilon$ fit in an α_1, α_2, or α_3 bin, so

$$A(L) \geq n + |\alpha_1| + |\alpha_2| + |\alpha_3|. \tag{2.3}$$

Let us assume that

$$\max\{r_1(n),\ r_2(n),\ r_3(n),\ r_4(n)\} < \frac{109}{71}. \tag{2.4}$$

Combining equations (2.1), (2.2), and (2.3), this tells us

$$\begin{aligned} \frac{n}{42} \cdot \frac{109}{71} &> |\alpha_1| + |\beta_1| \\ \frac{n}{6} \cdot \frac{109}{71} &> |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| \\ \frac{n}{2} \cdot \frac{109}{71} &> |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\alpha_3| + |\beta_3| \\ n \cdot \frac{109}{71} &> |\alpha_1| + |\alpha_2| + |\alpha_3| + n \end{aligned} \tag{2.5}$$

Because there are n pieces of size $\frac{1}{42} - 3\epsilon$, n of size $\frac{1}{7} + \epsilon$, and n of size $\frac{1}{3} + \epsilon$,

$$n = \sum_{b_w \in B} m_1$$

$$n = \sum_{b_w \in B} m_2 \tag{2.6}$$

$$n = \sum_{b_w \in B} m_3$$

From (2.6), we immediately have

$$-\frac{4}{42} n = -\frac{4}{42} \sum_{b_w \in B} m_1$$

$$-\frac{1}{2} n = -\frac{1}{2} \sum_{b_w \in B} m_2 \tag{2.7}$$

$$-n = -\sum_{b_w \in B} m_3$$

Summing equations (2.5) and (2.7),

$$\frac{109}{71} n\left(\frac{1}{42} + \frac{1}{6} + \frac{1}{2} + 1\right) - n\left(\frac{4}{42} + \frac{1}{2} + 1\right)$$

$$> 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3| + n \tag{2.8}$$

$$-\frac{4}{42} \sum_{b_w \in B} m_1 - \frac{1}{2} \sum_{b_w \in B} m_2 - \sum_{b_w \in B} m_3$$

Simplifying inequality (2.8) and rearranging terms:

$$\sum_{b_w \in B} \left(\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3\right) > 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|$$

$$\sum_{b_w \in \alpha_1} \left(\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3\right) + \sum_{b_w \in \beta_1} \left(\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3\right)$$

$$+ \sum_{b_w \in \alpha_2} \left(\frac{1}{2} m_2 + m_3\right) + \sum_{b_w \in \beta_2} \left(\frac{1}{2} m_2 + m_3\right) + \sum_{b_w \in \alpha_3} m_3 + \sum_{b_w \in \beta_3} m_3$$

$$> 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|. \qquad (2.9)$$

By considering separately each of the summations on the left hand side, we show that inequality (2.9) gives a contradiction.

(a) For $b_w \in \alpha_1$: $\frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 \leq 1$

$$\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 < 4$$

(b) For $b_w \in \beta_1$: $\frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 < \frac{1}{2}$

$$\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 < 2$$

(c) For $b_w \in \alpha_2$: $\frac{1}{7} m_2 + \frac{1}{3} m_3 \leq 1$

$$m_2 + 2\, m_3 \leq 6 + \frac{1}{7} m_2$$

Since the left hand side is an integer, $m_2 + 2m_3 \leq 6$

$$\frac{1}{2} m_2 + m_3 \leq 3$$

(d) For $b_w \in \beta_2$: $\frac{1}{7} m_2 + \frac{1}{3} m_3 < \frac{1}{2}$

$$\frac{1}{2} m_2 + m_3 < 2$$

(e) For $b_w \in \alpha_3$: $\frac{1}{3} m_3 < 1$

$$m_3 \le 2$$

(f) For $b_w \in \beta_3$: $\frac{1}{3} m_3 < \frac{1}{2}$

$$m_3 \le 1$$

Combining (a) - (f),

$$\sum_{b_w \in \alpha_1} (\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3) + \sum_{b_w \in \beta_1} (\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3)$$

$$+ \sum_{b_w \in \alpha_2} (\frac{1}{2} m_2 + m_3) + \sum_{b_w \in \beta_2} (\frac{1}{2} m_2 + m_3) + \sum_{b_w \in \alpha_3} m_3 + \sum_{b_w \in \beta_3} m_3$$

$$< 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|$$

This contradicts inequality (2.9). The assumption in (2.4) must be incorrect, from which we conclude that

$$\max \{\frac{A(L_1)}{OPT(L_1)}, \frac{A(L_1L_2)}{OPT(L_1L_2)}, \frac{A(L_1L_2L_3)}{OPT(L_1L_2L_3)}, \frac{A(L)}{OPT(L)}\} \ge \frac{109}{71} .$$

III. The Main Result

Define the sequence of integers $\{a_n\}$, for $n \geq 1$, by

$$a_1 = 2$$

$$a_{n+1} = 1 + \prod_{i=1}^{n} a_i \tag{3.1}$$

Thus, $\{a_n\} = \{ 2, 3, 7, 43, 1807, 3263443, \ldots\}$,

and notice that

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \ldots = 1.$$

This sequence has been studied by Golomb [10,11] and it is conjectured that the closest approximation to 1 from below, which is a sum of k reciprocal integers, is given by

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} = 1 - \frac{1}{a_{k+1}-1},$$

for every positive interger k.

In the proof of our lower bound result, we shall make use of the following simple lemma.

Lemma. Let $\{a_k\}$ be the sequence of integers defined above in (1). Then, for $1 \le k \le j$,

$$\frac{j+1}{a_k} \ge \frac{k}{a_k - 1}$$

Proof:

We first observe that

$$a_k \ge k+1$$

Then $$(k+1)a_k - (k+1) \ge k\, a_k$$

$$\frac{k+1}{a_k} \ge \frac{k}{a_k - 1}$$

and so, for $j \ge k$, $$\frac{j+1}{a_k} \ge \frac{k}{a_k - 1} . \quad \blacksquare$$

Motivated by the work in Section II, we now state and prove our main result.

Theorem. For any on-line one-dimensional packing algorithm A,

$$\lim_{n \to \infty} \left\{ \max_{OPT(L) = n} \frac{A(L)}{OPT(L)} \right\} \ge \frac{\sum_{i=1}^{t} \frac{i}{a_i - 1}}{\sum_{i=1}^{t} \frac{1}{a_i - 1}} > 1.5363$$

Proof:

For any positive integer $t \ge 3$, let ϵ be a small fixed number,

$$0 < \epsilon < \frac{1}{a_t(a_t - 1)(t-1)} .$$

We define pieces $p_1, \ldots, p_t$ to be of sizes

$$p_1 = \frac{1}{a_t - 1} - (t-1)\epsilon$$

and
$$p_j = \frac{1}{a_{t+1-j}} + \epsilon ,$$

for $2 \le j \le t$. Consider the list $L = L_1 L_2 \ldots L_t$, where each L_i consists of n pieces of size p_i, for n some multiple of $a_t - 1$. Then, for $1 \le k \le t$,

$$\mathrm{OPT}(L_1 L_2 \ldots L_k) = \frac{n}{a_{t+1-k} - 1} \tag{3.2}$$

and we can define the ratios

$$r_k(n) = \frac{A(L_1 L_2 \ldots L_k)}{\mathrm{OPT}(L_1 L_2 \ldots L_k)} . \tag{3.3}$$

We shall prove that

$$\max_{1 \le k \le t} \{r_k(n)\} \ge R_t , \tag{3.4}$$

where

$$R_t = \frac{\sum_{i=1}^{t} \frac{i}{a_i - 1}}{\sum_{i=1}^{t} \frac{1}{a_i - 1}} . \tag{3.5}$$

Let B denote the set of bins packed by an algorithm A, after the $(t-1)n$ pieces in list $L_1 L_2 \ldots L_{t-1}$ have been packed. Each bin $b_w \in B$ $(1 \le w \le |B|)$ contains $m_{i,w}$ pieces of size p_i, for all $1 \le i \le t-1$. For

notational convenience, we shall omit the double subscript and simply write m_i when we mean $m_{i,w}$. Note that $0 \le m_j < a_{t+1-j}$, for $1 \le j \le t-1$. For $1 \le k \le t-1$, the set α_k is defined to consist of those bins $b_w \in B$ which are at least half full and in which the smallest piece has size p_k. Similarly, we define β_k to be the set of bins $b_w \in B$ which are less than half full and in which the smallest piece has size p_k. So $|\alpha_k|(|\beta_k|)$ represents the number of bins in α_k (β_k), and, for $1 \le k \le t-1$

$$A(L_1 L_2 \dots L_K) = \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|). \tag{3.6}$$

Having packed $L_1 L_2 \dots L_{t-1}$, we note that it will not be possible to place any of the remaining n pieces of size p_t in any α_k bin. So we also have

$$A(L_1 L_2 \dots L_t) \ge n + \sum_{i=1}^{t-1} |\alpha_i|. \tag{3.7}$$

Let us assume that

$$\max_{1 \le i \le t} \{r_i(n)\} < R_t . \tag{3.8}$$

Making use of equations (3.2), (3.3), (3.6), and (3.7), this assumption leads to the following inequalities, for $1 \le k \le t-1$:

$$\begin{aligned} \frac{n}{a_{t+1-k}-1} \cdot R_t &> \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|) \\ n \cdot R_t &> n + \sum_{i=1}^{t-1} |\alpha_i| \end{aligned} \tag{3.9}$$

Because there are n pieces of each size p_i, we note that

$$n = \sum_{b_w \in B} m_{t-k+1}$$

for all k in the range $2 \le k \le t$. Thus,

$$-\frac{k}{a_k - 1} \cdot n = -\frac{k}{a_k - 1} \sum_{b_w \in B} m_{t-k+1} \tag{3.10}$$

Summing equations (3.9) and (3.10) over k gives

$$nR_t \sum_{k=1}^{t-1} \frac{1}{a_{t+1-k} - 1} + nR_t - n \sum_{k=2}^{t} \frac{k}{a_k - 1}$$

$$> \sum_{k=1}^{t-1} \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|) + n + \sum_{i=1}^{t-1} |\alpha_i| - \sum_{k=2}^{t} \frac{k}{a_k - 1} \sum_{b_w \in B} m_{t-k+1}$$

From (3.5), we observe that

$$R_t = \frac{1 + \sum_{k=2}^{t} \frac{k}{a_k - 1}}{1 + \sum_{k=1}^{t-1} \frac{1}{a_{t+1-k} - 1}}$$

and so inequality (3.11) can be simplified to give

$$\sum_{k=2}^{t} \frac{k}{a_k - 1} \sum_{b_w \in B} m_{t-k+1} > \sum_{k=1}^{t-1} \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|) + \sum_{i=1}^{t-1} |\alpha_i| \tag{3.12}$$

Inequality (3.12) further simplifies to give

$$\sum_{b_w \in B} \sum_{k=2}^{t} \frac{k}{a_k - 1} m_{t-k+1} > \sum_{j=1}^{t-1} ((j+1)|\alpha_{t-j}| + j|\beta_{t-j}|) \tag{3.13}$$

The remainder of this proof consists of showing that (3.13) gives a contradiction. In particular, we shall show that

$$\sum_{k=2}^{t} \frac{k}{a_k - 1} m_{t-k+1} \leq j + 1 \tag{3.14}$$

for any bin $b_w \in \alpha_{t-j}$ $(1 \leq j \leq t-1)$ and that

$$\sum_{k=2}^{t} \frac{k}{a_k - 1} m_{t-k+1} \leq j \tag{3.15}$$

for any bin $b_w \in \beta_{t-j}$ $(1 \leq j \leq t-1)$. From this we deduce that the assumption in (3.8) is incorrect, thereby proving the assertion of (3.4). The theorem follows immediately.

We first prove assertion (3.14). For $b_w \in \alpha_{t-j}$, then

$$p_1 m_1 + p_2 m_2 + \ldots + p_{t-1} m_{t-1} \leq 1 \tag{3.16}$$

and $p_{t-j} m_{t-j}$ is the first nonzero term. There are two cases.

(i) Assume that $j \leq t-2$. Then

$$\sum_{i=2}^{j+1} \frac{1}{a_i} m_{t-i+1} \leq 1$$

and

$$\frac{1}{a_{j+1} - 1} m_{t-j} + \sum_{i=2}^{j} \frac{1}{a_i} m_{t-i+1} \leq 1 + \frac{1}{a_{j+2} - 1} m_{t-j} \tag{3.17}$$

Recalling that $m_j < a_{t+1-j}$, then we know

$$m_{t-j} < a_{j+1} \tag{3.18}$$

Also, as a consequence of (3.1),

$$a_{j+2} - 1 = a_{j+1}(a_{j+1} - 1) \tag{3.19}$$

Using (3.18) and (3.19), inequality (3.17) gives

$$\frac{1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j} \frac{1}{a_i} m_{t-i+1} < 1 + \frac{1}{a_{j+1}-1} \tag{3.20}$$

From (3.1), we note that $a_{j+1} - 1$ is divisible by a_i, for all $i \le j$. Thus, the left hand side of (3.20) is a multiple of $\frac{1}{a_{j+1}-1}$, and we have

$$\frac{1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j} \frac{1}{a_i} m_{t-i+1} \le 1.$$

Thus,

$$\frac{j+1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j} \frac{j+1}{a_i} m_{t-i+1} \le j + 1 .$$

Applying the Lemma,

$$\frac{j+1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j} \frac{i}{a_i-1} m_{t-i+1} \le j + 1$$

and we have proved inequality (3.14) for $j \le t - 2$.

(ii) Assume that $j = t - 1$; i.e., $b_w \in \alpha_1$. Since $p_i > \frac{1}{a_{t+1-i}}$ for $2 \leq i \leq t - 1$, we conclude from (3.16) that

$$\left[\frac{1}{a_t - 1} - (t-1)\epsilon\right]m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} \leq 1.$$

Recalling how we chose ϵ,

$$\frac{1}{a_t-1} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < 1 + \frac{m_1}{a_t(a_t-1)} \tag{3.21}$$

Because $m_1 \leq a_t - 1$, the right hand side of (3.21) is less than $1 + \frac{1}{a_t}$. As in case (i), we also note that the left hand side of (3.21) is a multiple of $\frac{1}{a_t-1}$ and that $\frac{1}{a_t-1} > \frac{1}{a_t}$. Thus,

$$\frac{1}{a_t-1} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} \leq 1 \tag{3.22}$$

Similar to case (i), we multiply both sides of (3.22) by t and apply the Lemma in order to obtain the desired result:

$$\sum_{i=2}^{t} \frac{i}{a_i-1} m_{t-i+1} \leq t .$$

We now prove assertion (3.15). For $b_w \in \beta_{t-j}$, then

$$p_1 m_1 + p_2 m_2 + \dots + p_{t-1} m_{t-1} < \frac{1}{2}$$

and m_{t-j} is the first nonzero term. There are two cases.

(i) Assume that $j \le t-2$. Then

$$\sum_{i=2}^{j+1} \frac{1}{a_i} m_{t-i+1} < \frac{1}{2} \tag{3.23}$$

Multiplying both sides of (3.23) by $j+2$ and then applying the Lemma,

$$\sum_{i=2}^{j+1} \frac{i}{a_i - 1} m_{t-i+1} < \frac{j+2}{2} . \tag{3.24}$$

For $j \ge 2$, $\frac{j+2}{2} \le j$ and the result is proved. For $j = 1$, (3.24) reduces to $m_{t-1} < \frac{3}{2}$. Since m_{t-1} is an integer, this says $m_{t-1} \le 1$ and once again the desired result holds.

(ii) Assume that $j = t-1$; i.e., $b_w \in \beta_1$. Similar to inequality (3.21), we have

$$\frac{1}{a_t - 1} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < \frac{1}{2} + \frac{1}{a_t} \tag{3.25}$$

Multiplying both sides of (3.25) by t and applying the Lemma,

$$\sum_{i=2}^{t} \frac{i}{a_i - 1} m_{t-i+1} < \frac{t}{2} + \frac{t}{a_t}$$

For $t \ge 3$,

$$\frac{t}{a_t} < \frac{t-2}{2}$$

and so

$$\sum_{i=2}^{t} \frac{i}{a_i - 1} m_{t-i+1} < t-1$$

and the theorem is proved. ∎

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